

## Improved Control Charts to Monitor Changes in the Location Parameter

**Gadre M. P.**  
Shriram Institute of Information Technology,  
Paniv, Dist. Solapur , India-413113  
Email; [drmukund.gadre@gmail.com](mailto:drmukund.gadre@gmail.com)

**Rattihalli R. N.**  
27 SarnaikMal  
Samrat Nagar, Kolhapur India 416008  
Email: [ranganath.rattihalli@gmail.com](mailto:ranganath.rattihalli@gmail.com)

### Abstract

For univariate / multivariate processes, we propose an ‘Improved Control Chart to monitor the Location parameter’ (*ICC-L*). Looking at the comparative study based on illustrative examples considered, it is observed that under the *ATS* criterion, *ICC-L* performs uniformly better than the related charts and it is also better than the ‘Modified Control Chart with Warning Limits’ (*MCCWL*) to detect shifts in the location parameter. *ICC-L* may be used to monitor the production process.

**Keywords:** Control Limits, Warning Limits, Average Run Length, Average Time to Signal,

### 1. Introduction

In many manufacturing industries, for univariate production processes, control charts like,  $\bar{X}$ ,  $R$ ,  $S$ ,  $d$  and  $C$  charts are much useful to monitor the related location parameter. Also for multivariate processes, the charts like Hotelling  $T^2$  and  $\chi^2$  charts are useful to monitor the location parameter. For large shifts in the location parameter, such type of charts is quite effective. It is observed that, these charts are insensitive for small to moderate shifts.

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For univariate production processes, many researchers like Page (1954), Roberts (1959) have proposed control charts to monitor the process mean. At the beginning, ‘Average Run Length’ (*ARL*) model is used to develop the proposed chart. ‘Exponentially Weighted Moving Average’ (*EWMA*) was proposed by Roberts (1959). For moderate to large shifts in the process mean, Albin, Kang and Shea (1997) proposed the joint  $\bar{X}$ -*EWMA* chart that performs better as compared to the *EWMA* chart. Ncube (1990), Shamma and Shamma (1992) also developed such type of control charts.

Wu et al. (2001) introduced the term *ATS*, which is defined as the ‘Average number of units produced by the time the signal is observed. Effectiveness of a control chart can be measured by *ATS*. When charting statistic ( $\bar{X}$  here) is based on fixed sample size  $n$  (which can be determined under a suitable criterion),  $ATS = n(ARL)$ . The run length (*RL*), is the number of samples to get the signal, which depends on the stopping rule used to declare the signal and then to identify the assignable causes (if any). If stopping rule is to stop as soon as  $\bar{X}$  falls in the rejection region  $R$ , then will have waiting time distribution with parameter  $P$ , say, where  $P = P(\bar{X} \notin (LCL, UCL))$ . Also,  $ARL = \frac{1}{P}$ . Further,  $ATS = n(ARL) = \frac{n}{P}$ .

Under the *ATS* criterion, to determine design parameters of the chart, we consider a typical out of control state and minimize *ATS* of the out of control state (say  $ATS_I$ ) subject to *ATS* of in control state (say  $ATS_0$ ) is at least a fixed value  $\tau$ , say. (Refer to Wu et al. (2001) and Gadre and Rattihalli (2004)). The *ATS* model is given by,

$$\left. \begin{array}{l} \text{Minimize } ATS_I \\ \text{Subject to} \\ \\ ATS_0 \geq \tau. \end{array} \right\} \quad (I)$$

In Shewhart type control charts like  $(\bar{X}, R, np, c)$ , the decision depends only on the most recent value of the statistic. (One may refer to Montgomery (2008)). According to these charts process is in control if the charting statistic falls within the control limits. But it is evident that the degree of departure of the value of the statistic from the central line or the degree of closeness to control limits contains some information about the status of the process. An account of it is considered in charts with warning limits and in adaptive control charts. Rattihalli et al. (2021-2022) have developed 'Modified Control Chart With Warning Limits, (*MCCWL*) to monitor the location parameter. In *MCCWL* to detect shifts in the location parameter, the implementation is below. If the charting statistic ( $T$ , say) falls in the Rejection region or if two consecutive charting statistics lie in the warning region, the process has gone to the out of control status; otherwise, the process is running smoothly. With the help of numerical examples, it is observed that, *MCCWL* performs better as compared to the related charts.

To develop *ICC-L*, consider the following. We note that,

- (i) the value of a charting statistic  $T$  may fall outside the control limits, because either of the change in the location parameter (signal) or of the natural variability of the location parameter (false alarm).
- (ii) When a production process is to be started or recommenced, its quality, should not be taken as granted.

Keeping in view of the above, to assess the quality of the machine and to reduce the false alarm simultaneously, we propose the following scheme. Let  $T_r$  ( $r = 0, 1,$ ) be the charting statistic corresponding to the  $r^{th}$  sample. To reduce false alarms, we consider two consecutive charting statistics to lie beyond the control limits and to assess the quality of the machine at its

initial stage namely ‘signal is flagged if  $T_l$  lies beyond the control limits’. Based on this concept, we propose the following stopping rule.

**S: Stopping rule for the signal:** Stop if the charting statistic of the first sample ( $T_l$ , Say) falls outside control limits; otherwise for  $r > 1$ , if  $T_r$  and  $T_{r+1}$  fall outside the control limits. If it is assumed that  $T_0$  lie beyond the control limits, then this rule can be written as for  $r$  ( $r = 0, 1, \dots$ ), ‘Declare the process as out of control if two successive charting statistics fall outside the control limits.’

For example, an industrialist needs to purchase a machine to produce the products. Though the seller says that the machine is excellent, from the purchaser’s point of view, if he assumes that before purchasing (i.e. at time zero), the stopping rule mentioned above can be implemented. Considering the numerical examples, it is observed that the *ICC-L* is better than the related Shewhart type chart and *MCCWL*.

For a univariate process we consider the location parameter as the process mean and process fraction defective, for multivariate processes the process mean vector. Like *ICC-L*, the proposed related control charts are *ICC- $\bar{X}$* , *ICC-d* and *ICC- $\chi^2$* .

The proposed article is organized as below. Brief review of the *MCCWL* is given in **Section-2**. **Section-3** covers the basic notations and terms to design *ICC-L*. The operation, design and the *ATS* expression of *ICC-L* is derived in the same section. Additional notations of the *ICC -  $\bar{X}$*  are explained in **Section-4**. Also in the same section, numerical illustrations are considered to compare the performance of *ICC -  $\bar{X}$*  with the  $\bar{X}$  chart and *MCCWL -  $\bar{X}$* . **Section-5** covers the additional notations, the *ATS* expression, numerical illustrations and comparative study of the *ICC-d* with the *MCCWL-d* and the *np* chart. For multivariate processes, the notations and the

*ATS* expression of the  $ICC-\chi^2$  is given in **Section-6**. Numerical illustrations are considered and comparative study of  $ICC-\chi^2$  with the  $MCCWL-\chi^2$  and  $\chi^2$  chart are carried out in the same section. Next section covers the Concluding remarks.

## 2. Brief Review of *MCCWL*

### ‘Modified Control Chart with Warning Limits’ (*MCCWL*):

Rattihalli et al. (2022) developed a *MCCWL* to monitor location parameter of a production process. Location parameter may be the process mean, the number of defectives  $d$ , say in a sample of size  $n$  or the mean vector. Control charts related to the respective location parameters are named as  $MCCWL - \bar{X}$ ,  $MCCWL-d$  and  $MCCWL-\chi^2$ . These charts perform better as compared to the related Shewhart type control charts. In *MCCWL*, let  $A$ ,  $W$  and  $R$  be the acceptance, warning and the rejection regions respectively. Also, let  $k_l$  and  $k$  ( $k_l < k$ ) be the coefficients of warning and control limits respectively. Further, let ‘ $T_r$ ’ be the value of the charting statistic corresponding the  $r^{th}$  ( $r = 0, 1, ..$ ) sample. In *MCCWL*, it is assumed that  $T_r \in A$ . For  $r = 1, 2, \dots$ , the stopping rule for *MCCWL* is ‘Declare the process as out of control if  $T_r \in R$  or  $T_r \in W$  and  $T_{(r+1)} \in W$ . Let  $P_A = P(T \in A)$ . Similar meaning of the other two terms  $P_W$  and  $P_R$ . With the assumption of initial head start of *MCCWL*, the *ATS* expression is  $\frac{n_{mccw}(1+P_w)}{1-P_A(1+P_w)}$ . It is numerically illustrated that, *MCCWL* performs better as compared to the related Shewhart type control chart.

In the next section, the notations, operation and the derivation of the *ATS* expression of  $ICC-L$  are explained.

### 3. Basic Notations, the Operation, *ATS* criterion and Derivation of the *ATS* expression

Assume that the quality characteristic  $X$  has univariate / multivariate distribution ' $D$ , say' and the observations are independent. Following are the notations to be used in *ICC-L*

#### 3.1. Basic Notations:

1.  $n_{icc}$ : Sample Size.
2.  $\delta$ : Change in the location parameter.
3.  $M_0$ : In control location parameter.
4.  $\delta_l$ : Change in the location parameter, the magnitude of which is considered large enough to seriously impair the quality of the product.
5.  $\sigma$ : The process variability.
6.  $T$ : The charting statistic.
7.  $ARL(\delta)$ ,  $ARL_0 = ARL(0)$  and  $ARL_1 = ARL(\delta_l)$ : The related *ARLs*
8.  $ATS(\delta)$ :  $n_{icc}(ARL(M))$ . Similar mining for  $ATS_0$  and  $ATS_1$ ,
9.  $A$  and  $R$  are the acceptance and rejection regions respectively. Let ' $S$ ' be the '*Signal*' and is the absorbing state. Note that,  $R \subset S$ .
10.  $\tau$ : The minimum required value of  $ATS_0$ .

**3.2. The Operation:** In the following stepwise procedure of *ICC-L* is as below. Let *CNT* be the '*Counter*'.

**Step-I:** Initialize *CNT* to unity.

**Step-II:** Take a sample of size  $n_{icc}$ , find the charting Statistic  $T$ .

**Step-III:** If  $T \in R$ , increase  $CNT$  by unity and then go to the next step; otherwise, initialize  $CNT$  to zero and go back to **Step-II**.

**Step-IV:** If  $CNT = 2$ , go to the next step else go to **Step-II**.

**Step-V:** The process has gone out of control. Identify the assignable causes (if any). Make the corrective action and then restart the process. Move to **Step-1**.

**3.3.ARL expression**

As mentioned in **Sub-Section-3.1**, let  $P_A = P(T \in A)$  and  $P_R = P(T \in R)$  be the respective probabilities. Note that  $P_R = 1 - P_C$ .

By considering the state chain  $\{T_r, r=(0, 1, 2, \dots)\}$  will be a Markov chain with non absorbing states ‘A’ and ‘R’; and an absorbing state ‘RR’ (=Sig’) indicating the signal.

The transition probability matrix (t.p.m.) ‘P, say’ with all possible states (i.e. absorbing state and non absorbing states) of the chain  $\{T_r, r=(0, 1, 2, \dots)\}$  is

$$\mathbf{P} = \begin{matrix} & \begin{matrix} A & R & Sig \end{matrix} \\ \begin{matrix} A \\ R \\ Sig \end{matrix} & \begin{pmatrix} P_A & P_R & 0 \\ P_A & 0 & P_R \\ 0 & 0 & 1 \end{pmatrix} \end{matrix} \tag{2}$$

Here  $(T_0 \in R)$  is a head start and ‘R’ is the initial state. From Equation (2),  $\mathbf{P}_I$  is,

$$\mathbf{P}_I = \begin{matrix} & \begin{matrix} A & R \end{matrix} \\ \begin{matrix} A \\ R \end{matrix} & \begin{pmatrix} P_A & P_R \\ P_A & 0 \end{pmatrix} \end{matrix} \tag{3}$$

Now from Brooke and Evans (1972), the  $\underline{ARL}$  is given by ,

$$\underline{ARL} = (\mathbf{I} - \mathbf{P}_I)^{-1} \mathbf{1}$$

$$= \begin{bmatrix} \frac{1 + P_R}{1 - P_A(1 + P_R)} \\ \frac{1}{1 - P_A(1 + P_R)} \end{bmatrix} \quad (4)$$

As  $P_A + P_R = 1$ , it can be shown that,  $1 - P_A(1 + P_R) = (P_R)^2$ . Hence Equation (4) can be written as,

$$\underline{ARL} = \begin{bmatrix} \frac{1 + P_R}{P_R^2} \\ \frac{1}{P_R^2} \end{bmatrix}$$

As 'R' is the initial (or head start) state,  $ARL$  of  $ICC-L$  is  $ARL = \frac{1}{P_R^2}$  and  $ATS = n_{icc}(ARL)$ .

In the next section, for the quality characteristic  $X \sim N(\mu, \sigma^2)$  distribution, additional notations and the numerical illustrations of the  $ICC - \bar{X}$  are studied.

#### 4. $ICC - \bar{X}$



Here, the quality characteristic  $X \sim N(\mu, \sigma^2)$  distribution and the observations are independent. Note that  $\bar{X} \sim N(\mu, \sigma^2/n)$ . Following are some additional notations to be used in  $ICC - \bar{X}$ .

#### 4.1. Additional notations:

1.  $T$ :  $\bar{X}$ .
2.  $n_{icc\bar{x}}$ : Sample Size.
3.  $\mu$ : Process mean.
4.  $\mu_0$ : In-control process mean.
5.  $k_{icc\bar{x}}$ : The coefficient of the control limits.
6.  $LCL_{icc\bar{x}} = \mu_0 - k_{icc\bar{x}} \frac{\sigma}{\sqrt{n_{icc\bar{x}}}}$ ,  $UCL_{icc\bar{x}} = \mu_0 + k_{icc\bar{x}} \frac{\sigma}{\sqrt{n_{icc\bar{x}}}}$ , are the control limits.
7.  $A = (LCL_{icc\bar{x}}, UCL_{icc\bar{x}})$ ,  $R = (-\infty, LCL_{icc\bar{x}}) \cup (UCL_{icc\bar{x}}, \infty)$ .

As mentioned in the notations of  $ICC - \bar{X}$ ,

$$\begin{aligned}
 P_A(\delta) &= P(\bar{X} \in A) \\
 &= P\left\{\bar{X} \in (LCL_{icc\bar{x}}, UCL_{icc\bar{x}})\right\} \\
 &= P\left(\mu_0 - k_{icc\bar{x}} \frac{\sigma}{\sqrt{n_{icc\bar{x}}}} < \bar{X} < \mu_0 + \frac{k_{icc\bar{x}}\sigma}{\sqrt{n_{icc\bar{x}}}}\right) \\
 &= \varphi\left(k_{icc\bar{x}} + \delta\sqrt{n_{icc\bar{x}}}\right) - \varphi\left(-k_{icc\bar{x}} - \delta\sqrt{n_{icc\bar{x}}}\right)
 \end{aligned}$$

Similarly,  $P_R(\delta) = P(\bar{X} \in R) = 1 - P_A(\delta)$ . Note that the  $ATS$  expression is

$$ATS(\delta) = \frac{n_{icc\bar{x}}}{(P_R(\delta))^2} \quad (5)$$

Hence for given  $\delta$ ,  $ATS(\delta)$  is a function of the design parameters  $n_{icc\bar{x}}$  and  $k_{icc\bar{x}}$ . Let  $ATS_0 = ATS(0)$  and  $ATS_l = ATS(\delta_l)$ .  $ATS$  model given in Equation (1) is used to obtain the design parameters of  $ICC - \bar{X}$  with the help of  $ATS$  expression given in Equation (5). In the following, numerical illustrations are considered and the  $ATS$  performance of  $ICC - \bar{X}$  with the  $\bar{X}$  chart and  $MCCWL - \bar{X}$  is studied.

Next, we study the performance of  $ICC - \bar{X}$  with the  $\bar{X}$  chart and  $MCCWL - \bar{X}$  for some numerical illustrations.

#### 4.2. Some Examples of $ICC - \bar{X}$

**Example-1:** For  $\mu_0 = 0$ ,  $\sigma = 1$ ,  $\delta_l = 0.2$  and  $\tau = 2000$ , the design parameters and  $ATS_l$  values of  $ICC - \bar{X}$  and the two related charts are given below.

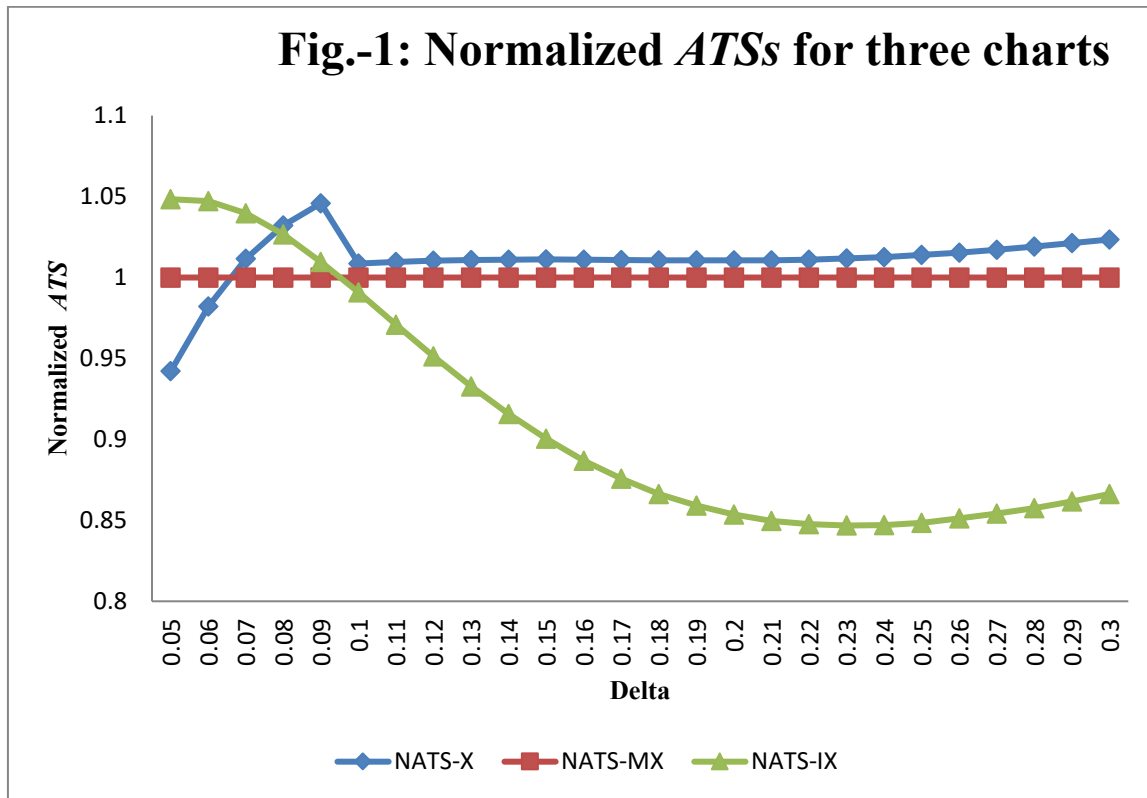
1. Shewhart  $\bar{X}$  chart:  $n_{\bar{x}} = 112$ ,  $k_{\bar{x}} = 1.91103$ ,  $ATS_l = 192.617$
2.  $MCCWL - \bar{X}$ :  $n_{mccw\bar{x}} = 107$ ,  $k_{1mccw\bar{x}} = 1.575$ ,  $k_{mccw\bar{x}} = 1.962$ ,  $ATS_l = 190.9142$
3.  $ICC - \bar{X}$ :  $n_{icc\bar{x}} = 97$ ,  $k_{icc\bar{x}} = 1.226$ ,  $ATS_l = 162.6760$

This example shows, not only  $ATS_l$  of  $ICC - \bar{X}$  is less than the  $\bar{X}$  chart and  $MCCWL - \bar{X}$ , but also  $n_{icc\bar{x}} < n_{mccw\bar{x}} < n_{\bar{x}}$ .

**Example-1 (Cont.):** To study the normalized  $ATS$  performance of the three related charts, for various values of  $\delta$  from 0.1 to 0.3,  $ATS(\delta)$  values are computed and are given in Table-1.

**Table-1:  $ATS$  values of  $\bar{X}$  chart,  $MCCWL-\bar{X}$  and  $ICC-\bar{X}$  chart for various values of  $\delta$** 

$\delta$ Value	$ATS$ of $\bar{X}$ Chart	$ATS$ of $MCCWL-\bar{X}$	$ATS$ of $ICC-\bar{X}$
0.05	1157.2	1228.2	1287.6
0.06	1029.2	1047.9	1097.4
0.07	902	891.6	926.9
0.08	784.2	759.7	779.9
0.09	679.9	650.1	656.4
0.1	564.501	559.665	554.5
0.11	489.889	485.229	471.1
0.12	428.327	423.941	403.3
0.13	377.410	373.361	348.2
0.14	335.159	331.478	303.5
0.15	299.968	296.656	267.1
0.16	270.544	267.583	237.3
0.17	245.852	243.204	213
0.18	225.058	222.678	192.9
0.19	207.492	205.327	176.4
0.2	192.613	190.607	162.7
0.21	179.983	178.079	151.3
0.22	169.244	167.388	141.9
0.23	160.103	158.244	134
0.24	152.318	150.410	127.4
0.25	145.690	143.690	121.9
0.26	140.051	137.925	117.4
0.27	135.261	132.979	113.6
0.28	131.200	128.740	110.4
0.29	127.769	125.113	107.8
0.3	124.879	122.018	105.7



## Results

- From Fig.-1 and Table-1, it is observed that, for  $\delta > 0.09$ ,  $ATS(\delta)$  of  $ICC-\bar{X}$  is smaller than the Shewhart  $\bar{X}$  chart and  $MCCWL-\bar{X}$ . Note that the proposed chart performs better than the other two charts, as for  $\delta > \delta_1$ ,  $ATS(\delta)$  of the  $ICC-\bar{X}$  is smaller than that of the related charts. Also, for  $\delta < \delta_1$ , which is far from in control  $\delta$ ,  $ATS(\delta)$  of  $ICC-\bar{X}$  is smaller than the other two charts. When  $\delta$  is not far from zero,  $ATS(\delta)$  of  $ICC-\bar{X}$  is larger than that of the related chart. From Fig.-1, it indicates that the proposed chart is uniformly superior to the related two control charts.

**Remark-1:** Result-1 indicates the proposed chart performs better as compared the remaining two charts.

**Example-2**

The 9 combinations of input parameters  $(\delta_l, \tau)$  are  $\{ (0.2, 2000), (0.5, 2000), (1, 2000), (0.2, 10000), (0.5, 10000), (1, 10000), (0.2, 50000), (0.5, 50000) \text{ and } (1, 50000) \}$

To study the effect on the performance of  $ICC-\bar{X}$  with the related two charts, using the  $ATS$  model, design parameters along with respective  $ATS_l$  values are computed and are given in Table-2.

**Table-2: Design parameters of Shewhart's  $\bar{X}$ ,  $MCCWL-\bar{X}$  and  $ICC-\bar{X}$  charts with respective  $ATS_l$  values**

Ex	$\delta_l$	$\tau$	$\bar{X}$ Chart			$MCCWL-\bar{X}$				$ICC-\bar{X}$		
			$n$	$k$	$ATS_l$	$n$	$k_l$	$k$	$ATS_l$	$n$	$k$	$ATS_l$
1	0.2	2000	112	1.91103	192.617	107	1.575	1.962	190.9142	97	1.226	162.6760
2	0.5	2000	32	2.40892	48.2964	28	1.905	2.51	47.2533	24	1.602	37.3445
3	1	2000	11	2.77621	15.590	10	2.175	2.85	15.1322	8	1.858	11.4993
4	0.2	10000	186	2.35344	287.984	167	1.835	2.455	282.2001	148	1.548	224.4742
5	0.5	10000	45	2.84082	64.6439	39	2.165	2.95	62.5997	33	1.9	47.3820
6	1	10000	14	3.19473	19.78	12	2.415	3.295	19.0275	10	2.15	14.0284
7	0.2	50000	269	2.78335	389.646	238	2.14	2.88	377.7186	196	1.862	287.3577
8	0.5	50000	59	3.24362	81.4096	51	2.435	3.345	78.2709	42	2.184	57.5073
9	1	50000	18	3.5681	23.9998	15	2.66	3.67	22.9871	12	2.422	16.5576

**Results:** For all 9 combinations the following results are observed and are mentioned below.

2.  $n_{icc\bar{x}} < n_{mccw\bar{x}} < n_{i\bar{x}}, ATS_{1icc\bar{x}} < ATS_{1mccw\bar{x}} < ATS_{1i\bar{x}}$
3.  $k_{icc\bar{x}} < k_{mccw\bar{x}} < k_{i\bar{x}}$

Thus, we conclude,  $ICC-\bar{X}$  performs better than the  $MCCWL-\bar{X}$  and  $\bar{X}$  chart. In the next section, 'ICC-d' is developed.

## 5. ICC-d

This section covers the additional notations, the *ATS* expression of *ICC-d* and the numerical illustrations related to *ICC-d*.

Here the quality characteristic  $X \sim \text{Ber}(p)$  distribution and the observations are independent. Note that  $np \sim b(n, p)$ . If  $n$  is a sample size,  $np = d$ , say. Following are some additional notations to be used in *ICC-d*.

### 5.1. Additional notations:

1.  $n_{iccd}$ : The sample size
2.  $T : d$ .
3.  $\delta : P$ : The fraction nonconforming.
4.  $M_0 : P_0$ : The in control fraction nonconforming.
5.  $\delta_1 : P_1$ : Design shift in the fraction nonconforming, the magnitude of which is considered large enough to seriously impair the quality of the product
6.  $ATS(P) : n_{iccp}(ARL(P))$ . Similar mining for  $ATS_0$  and  $ATS_1$ .
7.  $c_{iccd}$ : The acceptance number = Upper Control Limit of *ICC-d*.
8.  $A = (0, c_{iccd})$  and  $R = (c_{iccd}, \infty)$ .

**Remark-2:** For *ICC-d*, the charting statistic is  $d$ . Here,

$$P_a(P) = P(d \leq c_{iccd}) \text{ and } P_R(P) = P(d > c_{iccd}).$$

**Remark-3:** As given in Equation (5),  $ATS$  expression of  $ICC-d$  is

$$ATS(P) = \frac{n_{iccd}}{(P_R(P))^2} \quad (6)$$

Let  $ATS_0 = ATS(P_0)$  and  $ATS_1 = ATS(P_1)$ . From Equation (6) and the  $ATS$  model given in Equation (1), the design parameters of  $ICC - d$  are obtained. In the following, numerical illustrations are considered to study the performance of  $ICC - d$  with the  $d$  chart and  $MCCWL-d$ .

### 5.2. Numerical Illustrations of $ICC - d$

#### Example-3

For the input parameters  $p_0 = 0.01$ ,  $p_1 = 0.05$  and  $\tau = 10000$ , values of design parameters of the three related charts along with respective  $ATS_0$  and  $ATS_1$  are given below.

1.  **$d$  chart:**  $n_d = 77$ ,  $c_d = 3$ ,  $ATS_0 = 10152$ ,  $ATS_1 = 142.3002$
2.  **$MCCWL-d$ :**  $n_{mccwd} = 73$ ,  $c_{1mccwd} = 2$ ,  $c_{mccwd} = 3$ ,  $ATS_0 = 10120$ ,  $ATS_1 = 136.0062$
3.  **$ICC-d$ :**  $n_{iccd} = 41$ ,  $c_{iccd} = 1$ ,  $ATS_0 = 10187$ ,  $ATS_1 = 108.5905$

The above results clearly indicate that  $ATS_1$  of  $ICC - d$  are significantly less as compared to the  $d$  chart and  $MCCWL-d$ .

#### Example-4

Here the 18 combinations of the input parameters  $(p_0, p_1, \tau)$  are as follows.

$$P_0: 0.01, 0.005$$

$$p_1: 5p_0, 20p_0, 50p_0,$$

$$\tau : 10/p_0, 50/p_0, 100/p_0.$$

To study the *ATS* performance of *ICC-d* with the related two control charts is carried out. Following combinations of the input parameters are considered.

For these 18 combinations mentioned above, design parameters and the *ATS<sub>I</sub>* values for three control charts are obtained and are given in Table-3 below.

**Table-3: Design parameters of the *d* chart, *MCCWL-d* and *ICC-d* with respective *ATS<sub>I</sub>* values**

(p <sub>0</sub> , p <sub>1</sub> , τ)	<i>d</i> chart				<i>MCCWL-d</i>					<i>ICC-d</i>			
	<i>n</i>	<i>C</i>	<i>ATS<sub>0</sub></i>	<i>ATS<sub>I</sub></i>	<i>n</i>	<i>C<sub>I</sub></i>	<i>C</i>	<i>ATS<sub>0</sub></i>	<i>ATS<sub>I</sub></i>	<i>n</i>	<i>C</i>	<i>ATS<sub>0</sub></i>	<i>ATS<sub>I</sub></i>
(0.01, 0.05, 1000)	24	1	1006.1	70.7583	8	0	1	1017.3	67.6615	11	0	1004.2	59.1610
(0.01, 0.05, 5000)	41	2	5106	121.6084	35	1	2	5070.5	111.7982	54	1	5207.3	93.6857
(0.01, 0.05, 10000)	77	3	10152	142.3002	73	2	3	10120	136.0062	41	1	10187	108.5905
(0.01, 0.2, 1000)	9	1	2619	15.9633	5	0	1	1571.7	13.0979	6	0	1752	11.0207
(0.01, 0.2, 5000)	5	1	5101.3	19.0317	15	1	2	30005	23.2454	2	0	5050.4	15.4321
(0.01, 0.2, 10000)	17	2	27769	24.6242	15	1	2	30005	23.2454	15	1	161760	21.6238
(0.01, 0.5, 1000)	4	1	6756.4	5.8182	2	0	1	4186	4.8000	2	0	5050.4	3.5556
(0.01, 0.5, 5000)	4	1	6756.4	5.8182	6	1	2	277430	8.5622	2	0	5050.4	3.5556
(0.01, 0.5, 10000)	3	1	10067	6	6	1	2	277430	8.5622	6	1	2813100	7.5642
(0.005, 0.025, 2000)	47	1	2018.3	142.8090	16	0	1	2002.3	135.3059	22	0	2018	120.6214
(0.005, 0.025, 10000)	81	2	10152	245.3342	69	1	2	10135	225.9243	128	1	10120	184.7282
(0.005, 0.025, 20000)	153	3	20110	286.6547	145	2	3	20076	273.9367	82	1	20114	219.7465
(0.005, 0.1, 2000)	18	1	4963	32.7442	10	0	1	3053	26.8757	12	0	352102	23.3052
(0.005, 0.1, 10000)	9	1	10236	39.9717	3	0	1	10272	39.7323	4	0	10151	33.8217
(0.005, 0.1, 20000)	34	2	51047	50.4087	29	1	2	58910	47.5307	31	1	278090	44.9521
(0.005, 0.25, 2000)	7	1	13558	12.6114	4	0	1	7552.1	10.3388	4	0	10151	8.5598
(0.005, 0.25, 10000)	7	1	13558	12.6114	3	0	1	10272	10.6602	4	0	10151	8.5598
(0.005, 0.25, 20000)	5	1	20201	13.6170	12	1	2	41314	18.3735	2	0	20100	10.4490



**Results:**

4. For all the 18 combinations considered,  $ATS_{Iiccd} \leq ATS_{Imccwd}$ .
5. For 17 of the 18 combinations,  $ATS_{Iiccd} \leq ATS_{Imccwd} \leq ATS_{Id}$ . For the remaining single combination, to satisfy the  $ATS$  criterion for these three charts  $ATS_{0iccd}$  is significantly larger as compared to  $ATS_{0d}$ .

For multivariate processes, some Shewhart type multivariate control charts are  $\chi^2$  chart and  $T^2$  chart proposed by Hotelling (1947),  $ACS$  by Leoni and Costa (2017). In the next section, for multivariate normal distribution,  $ICC-\chi^2$  is developed,

## 6. $ICC-\chi^2$

Hotelling H. (1947) introduced the  $\chi^2$  and  $T^2$  control charts to monitor the mean vector of multivariate normal processes. Operations of these charts are like Shewhart type control charts. Therefore, for small and moderate shifts in the mean vector, these charts are insensitive. The proposed  $ICC-\chi^2$  is efficient as compared to the  $\chi^2$  chart. This section covers the notations and the  $ATS$  expression of  $ICC-\chi^2$ . Let  $p$  be the number of quality characteristics. For simplicity take  $p = 2$  and let  $X, Y$  be the two quality characteristics.

### 6.1. Additional Notations and the $ATS$ expression:

Following are the notations related to  $ICC-\chi^2$ .

1.  $n_{icc-ch}$ : The sample size

2.  $\underline{M}_0: \underline{\mu}_0 = (\mu_{0x}, \mu_{0y})$  : In control mean vector
3.  $\underline{M}_1: \underline{\mu}_1 = (\mu_{1x}, \mu_{1y})$ : Out of control mean vector
4.  $(\sigma_x, \sigma_y)$ : The process variability of the respective quality characteristics
5.  $\Sigma_0$ : The known covariance matrix of  $ICC-\chi^2$ , which depends on the correlation coefficient  $\rho$ .
6.  $\underline{\delta}_I = (\delta_{1x}, \delta_{1y}) = \left( \frac{\mu_{1x} - \mu_{0x}}{\sigma_{\bar{X}}}, \frac{\mu_{1y} - \mu_{0y}}{\sigma_{\bar{Y}}} \right)$ , where  $\sigma_{\bar{X}} = \frac{\sigma_x}{\sqrt{n}}$ ,  $\sigma_{\bar{Y}} = \frac{\sigma_y}{\sqrt{n}}$  for a bivariate process and it is shift in the standardized mean vector. Here,  $n$  is  $n_{icc-ch}$
7.  $\underline{\bar{X}}$  : Sample mean vector.
8.  $\chi^2 = n_{iwh} (\underline{\bar{X}} - \underline{\mu})' \Sigma_0^{-1} (\underline{\bar{X}} - \underline{\mu})$  : The test statistic for  $ICC-\chi^2$ .
9.  $\chi_0^2 = n_{iicc-ch} (\underline{\bar{X}} - \underline{\mu}_0)' \Sigma_0^{-1} (\underline{\bar{X}} - \underline{\mu}_0)$  : The test statistic for in control  $ICC-\chi^2$ .
10.  $\rho_{xy}$ : The related correlation coefficient .
11.  $k_{icc-ch}$ : The upper control limit of  $ICC-\chi^2$ .
12.  $A = (0, k_{icc-ch})$  and  $R = (k_{icc-ch}, \infty)$  are the acceptance and rejections respectively.
13.  $p_{0A} = p(\chi^2 \leq k_{icc-ch} | \underline{\mu} = \underline{\mu}_0)$  and  $p_{0R} = 1 - p_{0A}$  are the respective out of control probabilities.

**Remark-4:** *ATS* expression of  $ICC-\chi^2$  is,

$$ATS(\underline{\delta}) = \frac{n_{icc-ch}}{(P_R(\underline{\delta}))^2} \quad (7)$$

Next, for numerical illustrations considered,  $ATS$  performance of the  $ICC-\chi^2$  with the  $\chi^2$  chart and  $MCCWL-\chi^2$  is carried out.

## 6.2. Numerical Illustrations of $ICC-\chi^2$

### Example-5:

For  $\underline{\delta}_I = (0, 0.75)$ ,  $\rho = 0.7$  and  $\tau = 370$ . For these input parameters, the design parameters and  $ATS_I$  values of the  $\chi^2$  chart,  $MCCWL-\chi^2$  and  $ICC-\chi^2$  are given below.

(i)  $\chi^2$  chart:  $n_{ch} = 8$ ,  $k_{ch} = 7.67$ ,  $ATS_I = 12.3516$

(ii)  $MCCWL-\chi^2$ :  $n_{mcc-ch} = 8$ ,  $k_{1mccc-ch} = 5.01$ ,  $k_{mcc-ch} = 8.06$ ,  $ATS_I = 12.2129$

(iii)  $ICC-\chi^2$ :  $n_{icc-ch} = 7$ ,  $k_{icc-ch} = 3.97$ ,  $ATS_I = 9.8603$

This example shows that  $ATS_I$  of the  $ICC-\chi^2$  is less than  $ATS_I$  of the related three charts. Also  $n_{icc-ch} \leq n_{mcc-ch} \leq n_{ch}$ , which is an indication of reduction of the delay in detection of the shift.

### Example-6

Here, for  $\rho = 0.7$ ,  $\tau = 370$  and 14 combinations of  $\underline{\delta}_I$ , design parameters of the  $\chi^2$  chart,  $MCCWL-\chi^2$  and  $ICC-\chi^2$  are obtained along with  $ATS_I$  values of each of the combinations and are given in Table-4. Also, for  $\rho = 0.5$ ,  $\tau = 370$ , the design parameters and the input parameters along with the  $ATS_I$  values of the related three charts are given in Table-5.

**Table-4: Design parameters of the  $\chi^2$  chart,  $MCCWL\chi^2$  and  $ICC-\chi^2$  with respective  $ATS_I$  values**

$\delta_I = (\delta_{Ix}, \delta_{Iy})$	$\chi^2$ Chart			$MCCWL-\chi^2$				$ICC-\chi^2$		
	$\rho = 0.7$			$\rho = 0.7$				$\rho = 0.7$		
	$n$	$k$	$ATS_I$	$n$	$k_I$	$k$	$ATS_I$	$n$	$k$	$ATS_I$
(0, 0.50)	15	6.42	23.2314	14	5.01	6.66	23.0071	12	3.43	19.0532
(0, 0.75)	8	7.67	12.3516	8	5.01	8.06	12.2129	7	3.97	9.8603
(0, 1)	5	6.61	7.7828	5	6.01	8.82	7.6284	4	4.53	6.0764
(0, 1.50)	3	9.63	3.9810	3	7.01	9.76	3.9395	2	5.23	3.0438
(0.50, 0.50)	21	5.74	33.9704	20	4.01	6.12	33.7879	18	3.03	28.5184
(0.50, 0.75)	13	6.70	20.8782	13	5.01	6.84	20.6684	11	3.52	17.0359
(0.50, 1)	8	7.67	12.5848	8	5.01	8.06	12.4229	7	3.97	10.0321
(0.50, 1.50)	4	9.06	5.7212	4	6.01	9.37	5.6416	3	4.82	4.4269
(0.75, 0.75)	12	6.86	18.4397	11	5.01	7.24	18.2421	10	3.62	14.9806
(0.75, 1)	9	7.44	13.3028	8	5.01	8.06	13.1322	7	3.07	10.6222
(0.75, 1.50)	5	8.61	6.5748	4	6.01	9.37	6.4010	4	4.53	5.1406
(1, 1)	8	7.67	11.7290	7	4.01	8.44	11.6037	6	4.13	9.3409
(1, 1.50)	5	8.61	6.9357	4	6.01	9.37	6.8288	4	4.53	5.4161
(1.50, 1.50)	4	9.06	6.0804	4	6.01	9.37	5.9581	3	4.82	4.7287

**Table-5: Design parameters of the  $\chi^2$  chart,  $MCCWL\chi^2$  and  $ICC-\chi^2$  with respective  $ATS_I$  values**

$\delta_I = (\delta_{Ix}, \delta_{Iy})$	$\chi^2$ Chart			$MCCWL-\chi^2$				$ICC-\chi^2$		
	$\rho = 0.5$			$\rho = 0.5$				$\rho = 0.5$		
	$n$	$k$	$ATS_I$	$n$	$k_I$	$k$	$ATS_I$	$n$	$k$	$ATS_I$
(0, 0.50)	19	5.94	30.9906	19	4.01	6.26	30.8542	16	3.15	25.8896
(0, 0.75)	11	7.04	16.7419	10	5.01	7.48	16.5425	9	3.72	13.5130
(0, 1)	7	7.94	10.6136	7	6.01	8.05	10.4822	6	4.13	8.4243
(0, 1.50)	4	9.06	5.4827	3	6.01	10.13	5.4271	3	4.82	4.2280
(0.50, 0.50)	19	5.94	30.9906	19	4.01	6.26	30.8542	16	3.15	25.8896
(0.50, 0.75)	13	6.70	20.3251	12	5.01	7.03	20.1340	11	3.52	16.5768
(0.50, 1)	9	7.44	13.3533	8	5.01	8.06	13.1842	7	3.97	10.6660
(0.50, 1.50)	5	8.61	6.7608	4	6.01	9.37	6.6227	4	4.53	5.2821
(0.75, 0.75)	11	7.04	16.7419	10	5.01	7.48	16.5425	9	3.72	13.5130

(0.75, 1)	8	7.67	12.5371	8	5.01	8.06	12.3800	7	3.97	9.9969
(0.75, 1.50)	5	8.61	6.9472	4	6.01	9.37	6.84236	4	4.53	5.4250
(1, 1)	7	7.94	10.6136	7	6.01	8.05	10.4822	6	4.13	8.4243
(1, 1.50)	5	8.61	6.7608	4	6.01	9.37	6.6227	4	4.53	5.2821
(1.50, 1.50)	4	9.06	5.4827	4	6.01	8.05	5.3413	3	4.82	4.2280

**Results:** From Table-4, observe the following. For  $\delta_I \neq 0$ ,

- 6.  $n_{icc-ch} \leq n_{mccw-ch} \leq n_{ch}$
- 7.  $k_{icc-ch} < k_{ch}, k_{icc-ch} < k_{mccw-ch}$ .
- 8.  $ATS_I$  of  $ICC-\chi^2 \leq ATS_I$  of  $MCCWL-\chi^2 \leq ATS_I$  of  $\chi^2$  chart.

**Results:**

- 9. From Table-5, the results are same as mentioned in the Results 6 to 8.

This indicates that  $ICC-\chi^2$  detects shifts sooner. Also,  $UCL$  of  $\chi^2$  chart and  $MCCWL-\chi^2$  is smaller as compared to  $ICC-\chi^2$ .

### 7. Conclusions

The proposed  $ICC - \bar{X}$  performs much better as compared to  $MCCWL - \bar{X}$  and  $\bar{X}$  chart. For all combinations of the input parameters considered,  $ATS_{1icc\bar{x}} \leq ATS_{1mccw\bar{x}} \leq ATS_{1\bar{x}}$ . Also, for various  $\delta > 0.09$ ,  $ATS(\delta)$  of the  $ICC - \bar{X}$  is significantly less as compared to the  $\bar{X}$  chart and  $MCCWL - \bar{X}$ . Also, looking at the performance of  $ICC-d$ , for all 18 combinations considered  $ATS_{1iccd} \leq ATS_{1mccwd}$ . In the case of multivariate processes, for all 14 combinations of  $\delta_I$ ,  $ATS_{1icc-ch} \leq ATS_{1mcc-ch} \leq ATS_{1ch}$ .

Implementation/operation of the rule to stop the process for the *ICC-L* is simple. As these charts give a scope to account for doubtful situations, and decisions are taken by getting additional information, these charts seem to be used by the industries.

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